

Quadratic Expressions: Trading Land – Algebra

Overview: In this lesson students evaluate the fairness of a trade of land by comparing the lengths and areas of rectangles formed by adding the same number of meters to the length as subtracting from the width of a square.

Mathematics: To solve this task successfully, students need to decide on an organized way to compare the rectangles formed by modifying a square and making generalizations about the width, length, and area of the original square and the resulting rectangles.

Goals:

- Students will solve the problem using a variety of strategies.
- Students will develop an understanding of quadratic expressions—in expanded form and in factored form.
- Students will demonstrate the equivalence of expressions: $n^2 - 4$ and $(n + 2)(n - 2)$.
- Students will discuss the patterns that arise when comparing the side lengths of the squares and rectangles and they will write symbolic representations of those relationships.
- Students will graph the relationships of the side lengths and the areas of the rectangles and describe their shapes.
- Students will justify their solutions to the problem.

Algebra Standards:

- *4.0 Students simplify expressions prior to solving linear equations and inequalities in one variable, such as $3(2x-5)+4(x-2)=12$
- *10.0 Students add, subtract, multiply and divide monomials and polynomials. Students solve multistep problems, including word problems, by using these techniques.
- 11.0 Students apply basic factoring techniques to second- and simple third-degree polynomials. These techniques include finding a common factor for all terms in a polynomial, recognizing the difference of two squares, and recognizing perfect squares of binomials.
- 17.0 Students determine the domain of independent variables and the range of dependent variables defined by a graph, a set of ordered pairs, or a symbolic expression.
- *23.0 Students apply quadratic equations to physical problems, such as the motion of an object under the force of gravity.

Building on Prior Knowledge:

- *7.0 Students verify that a point lies on a line, given the equation of the line.

Materials: Trading Land Task (attached); chart paper; graph paper; markers

Note: Developing an understanding of the mathematical concepts and skills embedded in a standard requires having multiple opportunities over time to engage in solving a range of different types of problems which utilize the concepts or skills in question.

Phase	Action	Comments
S E T U P	Prior to the lesson: • arrange the desks so that students are in groups of 4. • determine student groups prior to the lesson so that students who complement each other's skills and knowledge core are working together. • place materials for the task at each grouping. • solve the task yourself. <u>HOW DO I SET UP THE LESSON?</u> Ask students to follow along as you read the situation presented in the Trading Land Task. Introduce the task by asking students to think about the trade that U.S. Mall, Inc. offered and answer the first question (Does it seem like a fair trade?). Then have several students explain to the class what they are trying to find when solving the problem. Stress to students that they will be expected to explain how and why they solved the problem a particular way and to refer to the context of the problem.	Students will be more successful in this task if they understand what is expected in terms of group work and the final product. It is critical that you solve the problem in as many ways as possible so that you become familiar with strategies students may use. This will allow you to better understand students' thinking. As you read through this lesson plan, different strategies for solving the problem will be given. <u>HOW DO I SET UP THE LESSON?</u> As students describe the task, listen for their understanding of the goals of the task. It is important that they indicate the goal is to determine and demonstrate how to compare the side lengths of the rectangles and their areas to evaluate if the trade is a fair trade. Make sure they understand the condition of the trade: one dimension of the rectangle is two meters longer than a side of the square, the other dimension is 2 meters shorter than a side of the square. Be careful not to tell students how to solve the task or to set up a procedure for solving the task because your goal is for students to do the problem solving.

Phase	Action	Comments
E X P L O R E	<u>PRIVATE PROBLEM-SOLVING TIME</u> Give students 5 - 7 minutes of private think time to begin to solve the problem individually. Circulate among the groups assessing students' understanding of the idea below.	<u>PRIVATE PROBLEM-SOLVING TIME</u> Make sure that students' thinking is not interrupted by the talking of other students. If students begin talking, tell them that they will have time to share their thoughts in a few minutes.
	<u>FACILITATING SMALL-GROUP PROBLEM SOLVING</u> As you circulate among the groups, press students to think about systematic ways to compare the rectangles. After explaining their initial thinking, you might say, “What comparisons can you make to evaluate whether this is a fair trade? How can you compare the area of the new lot with the area of the original square?”	<u>FACILITATING SMALL-GROUP PROBLEM SOLVING</u> The teacher's role when students are working in small groups is to circulate and listen with the goal of understanding students' ideas and asking questions that will advance student work.
	<u>What do I do if students have difficulty getting started?</u> Allow students to work in their groups to solve the problem. Assist students/groups who are struggling to get started by prompting with questions such as: – What would be a fair trade? – How could you tell if a square and the resulting rectangle are the same? – What comparisons can you make? – What side lengths have you tried? Is there a pattern? If so, can you describe it? – What can you do to test different cases? – How can you find the area of the two plots of land (the original square and the resulting rectangle?)?	<u>What do I do if students have difficulty getting started?</u> By asking a question such as, “What is the problem asking me to do? Have I solved a similar problem before?”, the teacher is providing students with a question that can be used over and over when problem solving. This will help them focus on what they know, what they were given, and what they need to determine.

Phase	Action	Comments
E X P L O R E (Cont'd)	<u>What misconceptions might students have?</u> Look for and clarify any misconceptions students may have. <i>Not understanding that the perimeter remains constant. What is changing and what remains the same?</i> <i>Confusing m with a variable. Remind them that in this case m stands for meters and m^2 stands for square meters.</i> <i>Adding to same numbers of meters to one side as reducing the same number of meters from the other results in the resulting shape having the same area. What is the perimeter of the resulting rectangle? What is the area of the resulting rectangle? Do you see a pattern?</i> <i>Some students may think that $(n + 2)(n - 2)$ is equivalent to $n + 2 \times n - 2$. Ask students to substitute some values for n to verify that these are not equivalent. Some students may write the area of the new rectangle as $n + 2 \times n - 2$.</i>	<u>What misconceptions might students have?</u> Misconceptions are common. Some strategies for helping students gain a better understanding include: – Ask students to use other representations of the same problem and find connections among them.

Phase	Action	Comments
E X P L O R E (Cont'd)	<u>Which problem-solving strategies might be used by students?</u> <u>How do I advance students' understanding of mathematical concepts or strategies when they are working with each strategy?</u> Students will approach the problem using a variety of strategies. Some strategies are shown below. Questions for assessing understanding and advancing student learning are listed for each. <u>Using Drawings</u> I. <i>showing the side length and area of the original square and the length and width of the new rectangle</i> What happens to the perimeter of the new lot? What happens to the area? If the side length of the square is n, what is the length of the rectangle? What is the width of the rectangle? What is the area of the rectangle? ----- <u>Using a Table</u> What patterns do you see on the table? Can we use those patterns to predict the numbers in the next row? Can we find a general expression for each column?	<u>Which problem-solving strategies might be used by students?</u> <u>How do I advance students' understanding of mathematical concepts or strategies when they are working with each strategy?</u> <u>Using Drawings</u> If students are drawing many squares and rectangles, press them to think of a more efficient and systematic way to record their findings. You may want to ask them what they think is a good way to organize information, or how they have organized information in past problems. ----- <u>Using a Table</u> See Appendix B.

Phase	Action	Comments
SHARE DISCUSS AND ANALYZE	<u>FACILITATING THE GROUP DISCUSSION</u> <i>What order will I have students post solution paths so I will be able to help students make connections between the solution paths?</i> As you circulate among the groups, look for solutions that will be shared with the whole group and consider the order in which they will be shared. Ask students to explain their solutions to you as you walk around. Make certain they can make sense of their solutions in terms of their representations. For this task, it would useful to show a table in which the general rule of the area of the rectangle is written as $n^2 - 4$ and another one written as $(n + 2)(n - 2)$. Ask students to post their work in the front of the classroom. <i>What question can I ask throughout the discussion that will help students keep the context and the goal of the problem in mind? (Driving Questions*)</i>	<u>FACILITATING THE GROUP DISCUSSION</u> <i>What order will I have students post solution paths so I will be able to help students make connections between the solution paths?</i> Even though you may display all solution paths, you should strategically pick specific solution paths to discuss with the whole group. <i>What question can I ask throughout the discussion that will help students keep the context and the goal of the problem in mind? (Driving questions*)</i> * Driving Questions have been provided because they will help to stimulate student interest, maintain the focus of the discussion on the problem context, and focus the discussion on key mathematical ideas. Many of the questions require students to take a position or to wonder about mathematical ideas or problem-solving strategies.

Phase	Action	Comments
S H A R E D I S C U S S A N D A N A L Y Z E (Cont'd)	<u>Solutions using drawings</u> Ask students to present their solutions and explain how their drawings helped them to evaluate the trade. How did you use the pictures to decide if the trade was fair? You might ask, “Were the pictures useful in seeing whether the lots were the same?”	<u>Accountable Talk</u> <u>Pointing to the drawing</u> Asking student to point to the drawing and showing where the evidence for their reasoning is.
	<u>Solutions using a table</u> What patterns do you see in the table? Are the expressions $n^2 - 4$ and $(n + 2)(n - 2)$ equivalent? How do you know? Do they have the same value for a given n?	<u>Repeating or Paraphrasing Ideas</u> Ask other students to put explanations given by their peers into their own words. This is a means of assessing understanding and providing others in the class with a second opportunity to hear the explanation.
	Could we check whether the two expressions are equivalent by graphing $A = n^2 - 4$ and $A = (n + 2)(n - 2)$?	<u>Position-Driven Discussion</u> Press students to take a position and to support their claims with evidence from the drawings. Students must say why their drawings represent that fraction of the pan of brownies. In doing so they will have to provide reasons for their claims.
	What would the graph of the width, $n - 2$, and of the length, $n + 2$, of the rectangle look like?	
	Graph the equation for the relationship between the side length and the area of the original lot. Use n values from -10 to 10 and A values from -20 to 100.	
	Write an equation for the relationship between the side length of the original lot and the area of the new lot. Graph the equation using the same window settings as before.	
	Describe the shapes of your graphs.	
	Do all the values for area and side length on your graph make sense? Why or why not?	

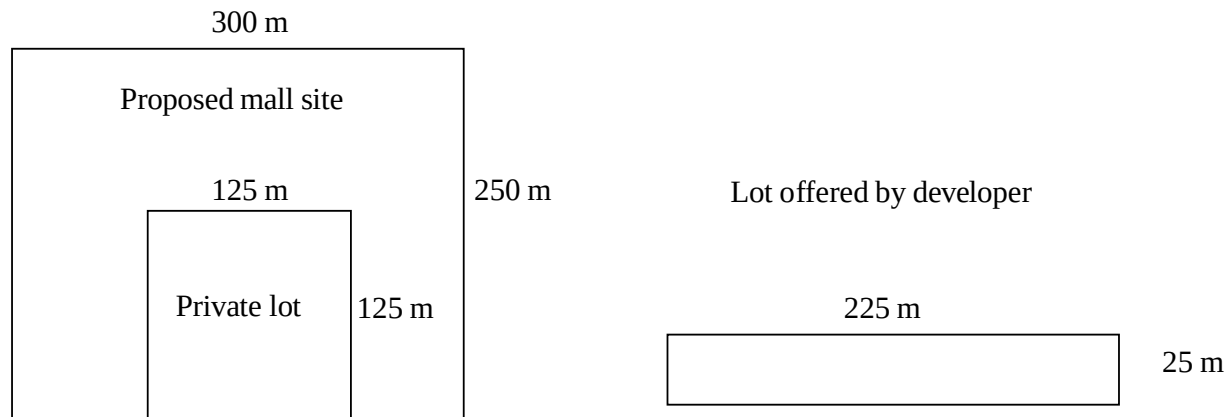
Quadratic Expressions: Trading Land

Your task: In this problem you will explore whether the trade offered will always be fair, sometimes be fair, or never be fair. Read the situation and develop a way to evaluate whether the proposed trade is a fair trade for any side length.

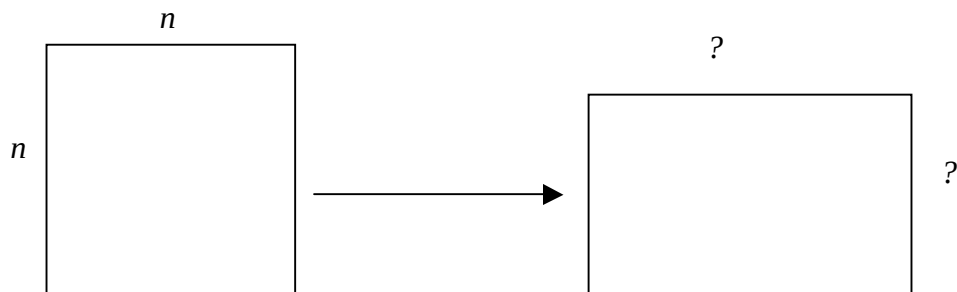
If you gave a friend two \$1 dollar bills, and your friend gave you eight quarters, you would consider it a fair trade. If you left your job an hour early one day and worked an hour later the next day, your boss would probably consider it a fair trade. Sometimes, it is not easy to determine whether a trade is fair.

Think about this!

U.S. Mall, Incorporated, wants to build a new shopping center. The mall developer has bought all the land on the proposed site except for one square lot that measures 125 meters on each side. The family that owns the lot is reluctant to sell it. In exchange for the lot, the developer has offered to give the family a rectangular lot that is 100 meters longer on one side and 100 meters shorter on another side than the square lot. Do you think this is a fair trade? Why or why not?



Suppose you own a square piece of land with sides n meters long. You trade your land for a rectangular lot. The length of your new lot is 2 meters longer than the side length of your original lot, and the width of your new lot is 2 meters shorter than the side length of the original lot.



- A. For which side lengths, if any, is this a fair trade? Use pictures, tables, graphs, symbols, and/or words to show how you know whether this is a fair trade for any side length.

B. Copy and complete the table below.

Original square		New rectangle			Difference in areas (m ²)
Side length (m)	Area (m ²)	Length (m)	Width (m)	Area (m ²)	
3	9	5	1	5	4
4					
5					
6					
7					

C. For each side length in the table, tell how the area of your new lot compares with the area of the original lot.

D. The side length of the original square lot was n meters. For each column in the table, write an expression for the values in the column in terms of n . For example, the expression for the area of the original square is n^2 .

Solutions for the table.

Original square		New rectangle			Difference in areas (m ²)
Side length (m)	Area (m ²)	Length (m)	Width (m)	Area (m ²)	
3	9	5	1	5	4
4	16	6	2	12	4
5	25	7	3	21	4
6	36	8	4	32	4
7	49	9	5	45	4
100	10000	102	98	9996	4
n	n^2	$n + 2$	$n - 2$	$(n + 2)(n - 2)$	4

Possible Responses for the Share, Discuss and Analyze Phase

The trade was not fair because the area of the new lot is always smaller than the area of the original lot. This was not easy to see in the pictures.

Solutions using a table

Patterns

- There is always a difference of 4 m between the length and width of the new rectangle.
- The length and width of the new rectangle have a sum that is twice the length of the side of the square.
- The area of the new rectangle is always less than the area of the original square.
- The area of the new rectangle is always 4 m^2 less than the area of the square.

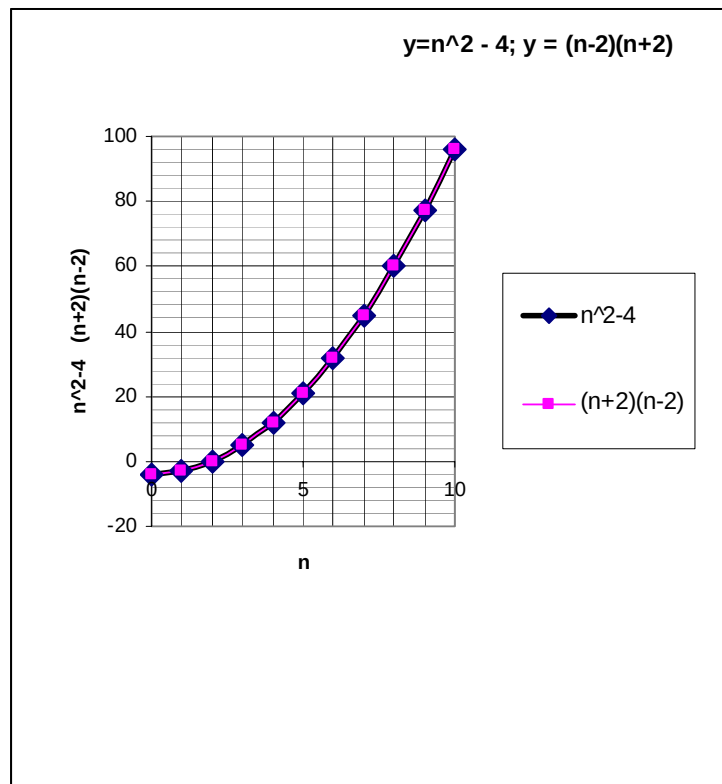
Equivalence of $n^2 - 4$ and $(n+2)(n-2)$

- If you subtract 4 from the “Area” of the original square column, which is $n^2 - 4$, you always get the same as the “Area” of the new rectangle column which is $(n+2)(n-2)$.

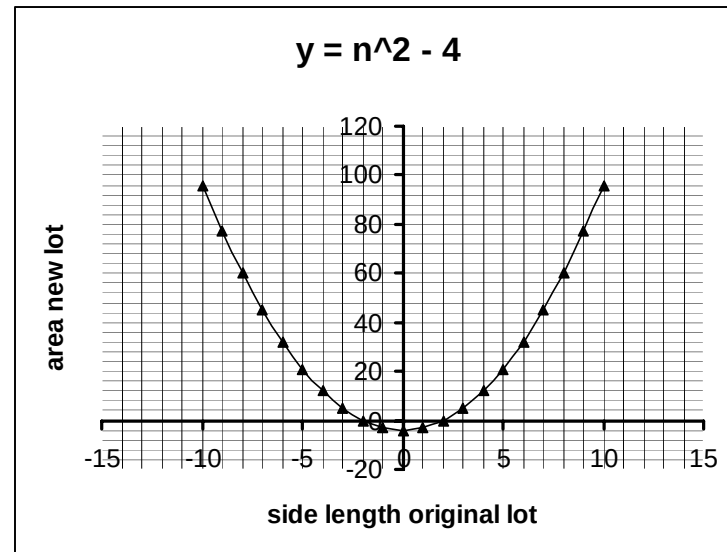
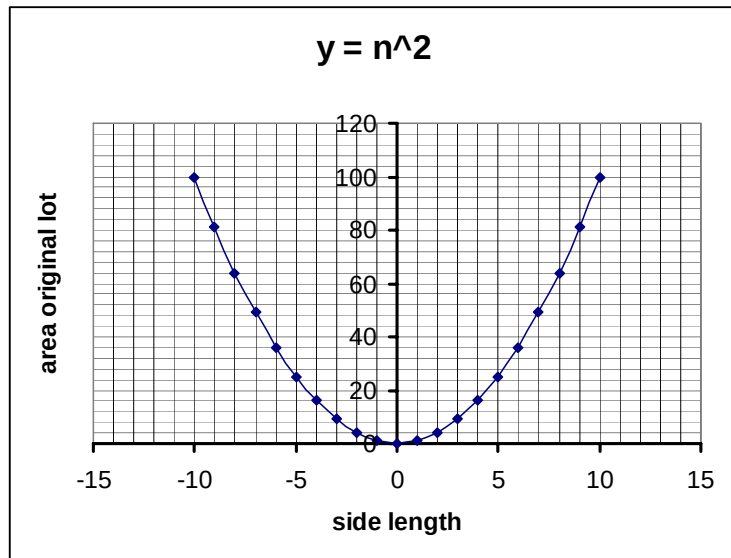
Solutions using a graph

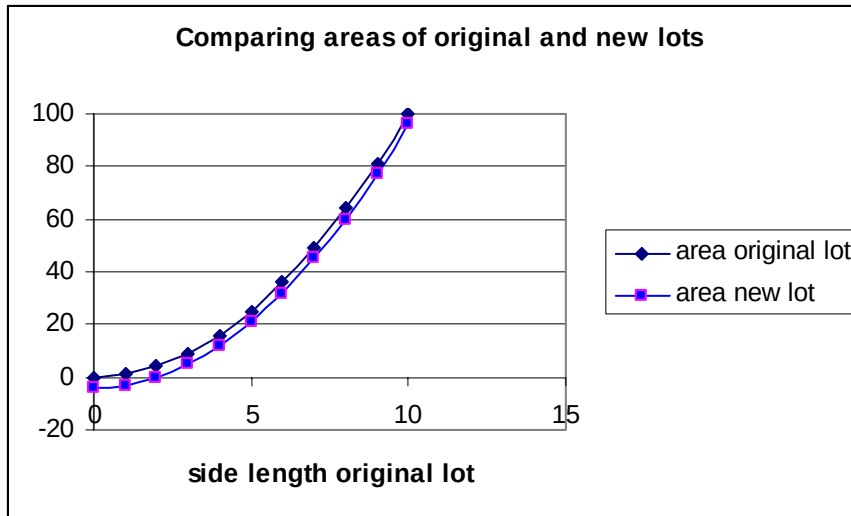
Equivalence of n^2-4 and $(n+2)(n-2)$

Students might graph $y = n^2-4$ and $y = (n+2)(n-2)$ on graph paper or using a graphing calculator. This will illustrate the equivalence of the 2 expressions since both graphs are the same.



Graphs of side length and areas of original lot and new lot





- The graphs show that the trade is not fair. For any given side length value, the area value of the original lot is always higher (more than) the area of the new lot.
- Both graphs are parabolas because they are graphs of quadratic functions.
- The values of side lengths below 0 for the original lot and below 2 for the new lot do not make sense because they would result in negative lengths for the sides of the lots.